

ON THE NON-VANISHING CONJECTURE AND EXISTENCE OF LOG MINIMAL MODELS

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ABSTRACT. We prove that the non-vanishing conjecture and the log minimal model conjecture for projective log canonical pairs can be reduced to the non-vanishing conjecture for smooth projective varieties such that the boundary divisor is zero.

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1. INTRODUCTION

Throughout this paper we will work over the complex number field, and we denote Conjecture \bullet with $\dim X = n$ (resp. $\dim X \leq n$) by Conjecture \bullet_n (resp. Conjecture $\bullet_{\leq n}$).

In this paper we deal with the following two conjectures.

Conjecture 1.1 (Non-vanishing). *Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is pseudo-effective, then there is an effective \mathbb{R} -divisor D such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

Conjecture 1.2 (Existence of log minimal model). *Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a log minimal model.*

Birkar [B1] proved that Conjecture 1.1_n implies Conjecture 1.2_n. On the other hand, Gongyo [G] proved that Conjecture 1.1_n for Kawamata log terminal pairs with boundary \mathbb{Q} -divisors implies Conjecture 1.1_n for log canonical pairs with boundary \mathbb{R} -divisors assuming the abundance

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conjecture for d -dimensional log canonical pairs with $d \leq n-1$. Today Conjecture 1.1_{≤3} and Conjecture 1.2_{≤4} is proved (Conjecture 1.2₄ was first proved by Shokurov [S] and another proof was given by Birkar [B2]) but Conjecture 1.1 and Conjecture 1.2 are still open in higher dimension.

In this paper we study the relation between the above two conjectures and the following special case of Conjecture 1.1.

Conjecture 1.3 (Non-vanishing for smooth varieties). *Let X be a smooth projective variety. If K_X is pseudo-effective, then there is an effective \mathbb{Q} -divisor D such that $K_X \sim_{\mathbb{Q}} D$.*

The following theorem is the main result of this paper.

Theorem 1.4. *Conjecture 1.3_n implies Conjecture 1.1_{≤n} and Conjecture 1.2_{≤n}.*

We remark that in Theorem 1.4 we do not have any assumptions about the abundance conjecture. The proof of Theorem 1.4 heavily depends on the arguments in [H]. In [H] all arguments were carried out in the framework of log canonical pairs with boundary \mathbb{Q} -divisors, but in this paper the boundary divisor of any log canonical pair may be an \mathbb{R} -divisor.

From Theorem 1.4 we immediately obtain the following corollaries.

Corollary 1.5. *Conjecture 1.1_n and Conjecture 1.3_n are equivalent.*

Corollary 1.6. *Conjecture 1.3_n implies Conjecture 1.2_n.*

Corollary 1.5 is a generalization of [G, Theorem 1.5] and [DHP, Theorem 8.8], and Corollary 1.6 is a generalization of [B1, Theorem 1.4]. We emphasize that by Corollary 1.6 we can reduce the log minimal model conjecture for log canonical pairs to the non-vanishing conjecture in very simple situation.

The contents of this paper are the following. In Section 2, we collect some notations and definitions, and we recall two important theorems (cf. Theorem 2.7 and Theorem 2.8). In Section 3 we prove Theorem 1.4, Corollary 1.5 and Corollary 1.6.

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2. PRELIMINARIES

In this section we collect some notations and definitions. We will freely use the notations and definitions in [BCHM] and [H]. For basic definitions of divisors, see [H].

2.1 (Maps). Let $f : X \rightarrow Y$ be a morphism of normal projective varieties. Then f is a *contraction* if f is surjective and it has connected fibers.

Let $f : X \dashrightarrow Y$ be a birational map of normal projective varieties. Then f is a *birational contraction* if f^{-1} does not contract any divisors. Let D be an \mathbb{R} -divisor on X . Unless otherwise stated, we mean f_*D by denoting D_Y .

2.2 (Singularities of pairs). A *pair* (X, Δ) consists of a normal projective variety X and a boundary \mathbb{R} -divisor, that is, an \mathbb{R} -divisor whose coefficients belong to $[0, 1]$, on X such that $K_X + \Delta$ is \mathbb{R} -Cartier.

Let (X, Δ) be a pair and $f : Y \rightarrow X$ be a log resolution of (X, Δ) . Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i$$

where E_i are prime divisors on Y and $a(E_i, X, \Delta)$ is a real number for any i . Then we call $a(E_i, X, \Delta)$ the *discrepancy* of E_i with respect to (X, Δ) . The pair (X, Δ) is called *Kawamata log terminal* (*klt*, for short) if $a(E_i, X, \Delta) > -1$ for any log resolution f of (X, Δ) and any E_i on Y . (X, Δ) is called *log canonical* (*lc*, for short) if $a(E_i, X, \Delta) \geq -1$ for any log resolution f of (X, Δ) and any E_i on Y . (X, Δ) is called *divisorial log terminal* (*dlt*, for short) if Δ is a boundary \mathbb{R} -divisor and there exists a log resolution $f : Y \rightarrow X$ of (X, Δ) such that $a(E, X, \Delta) > -1$ for any f -exceptional prime divisor E on Y .

Next we introduce the definition of some models. For some remarks of the models, see [H, Remark 2.7].

Definition 2.3 (Weak lc models and log minimal models, cf. [B3, Definition 2.1], [H, Definition 2.5]). Let (X, Δ) be a log canonical pair and $\phi : X \dashrightarrow X'$ be a birational map to a normal projective variety X' . Let E be the reduced ϕ^{-1} -exceptional divisor on X' , that is, $E = \sum E_j$ where E_j are ϕ^{-1} -exceptional prime divisors on X' . Then the pair $(X', \Delta' = \phi_*\Delta + E)$ is called a *log birational model* of (X, Δ) . A log birational model (X', Δ') of (X, Δ) is a *weak log canonical model* (*weak lc model*, for short) if

- $K_{X'} + \Delta'$ is nef, and

- for any prime divisor D on X which is exceptional over X' , we have

$$a(D, X, \Delta) \leq a(D, X', \Delta').$$

A weak lc model (X', Δ') of (X, Δ) is a *log minimal model* if

- X' is \mathbb{Q} -factorial, and
- the above inequality on discrepancies is strict.

A log minimal model (X', Δ') of (X, Δ) is called a *good minimal model* if $K_{X'} + \Delta'$ is semi-ample.

Definition 2.4 (Mori fiber spaces, cf. [B3, Definition 2.2], [H, Definition 2.5]). Let (X, Δ) be a log canonical pair and (X', Δ') be a log birational model of (X, Δ) .

Then (X', Δ') is called a *Mori fiber space* if X' is \mathbb{Q} -factorial and there is a contraction $X' \rightarrow W$ with $\dim W < \dim X'$ such that

- the relative Picard number $\rho(X'/W)$ is one and $K_{X'} + \Delta'$ is anti-ample over W , and
- for any prime divisor D over X , we have

$$a(D, X, \Delta) \leq a(D, X', \Delta')$$

and strict inequality holds if D is a divisor on X and exceptional over X' .

Finally we introduce the definition of log canonical thresholds and pseudo-effective thresholds, and two important theorems which are proved by Hacon, McKernan and Xu [HMX].

Definition 2.5 (Log canonical thresholds, cf. [HMX]). Let (X, Δ) be a log canonical pair and let $M \neq 0$ be an effective \mathbb{R} -Cartier \mathbb{R} -divisor. Then the *log canonical threshold* of M with respect to (X, Δ) , denoted by $\text{lct}(X, \Delta; M)$, is

$$\text{lct}(X, \Delta; M) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is log canonical}\}.$$

Definition 2.6 (Pseudo-effective thresholds). Let (X, Δ) be a log canonical pair and M be an effective \mathbb{R} -Cartier \mathbb{R} -divisor such that $K_X + \Delta + tM$ is pseudo-effective for some $t \geq 0$. Then the *pseudo-effective threshold* of M with respect to (X, Δ) , denoted by $\tau(X, \Delta; M)$, is

$$\tau(X, \Delta; M) = \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tM \text{ is pseudo-effective}\}.$$

Theorem 2.7 (ACC for log canonical thresholds, cf. [HMX, Theorem 1.1]). Fix a positive integer n , a set $I \subset [0, 1]$ and a set $J \subset \mathbb{R}_{>0}$, where I and J satisfy the DCC. Let $\mathfrak{T}_n(I)$ be the set of log canonical pairs

(X, Δ) , where X is a variety of dimension n and the coefficients of Δ belong to I . Then the set

$$\{\text{lct}(X, \Delta; M) \mid (X, \Delta) \in \mathfrak{T}_n(I), \text{ the coefficients of } M \text{ belong to } J\}$$

satisfies the ACC.

Theorem 2.8 (ACC for numerically trivial pairs, cf. [HMX, Theorem D]). *Fix a positive integer n and a set $I \subset [0, 1]$, which satisfies the DCC.*

Then there is a finite set $I_0 \subset I$ with the following property:

If (X, Δ) is a log canonical pair such that

- (i) X is projective of dimension n ,*
- (ii) the coefficients of Δ belong to I , and*
- (iii) $K_X + \Delta$ is numerically trivial,*

then the coefficients of Δ belong to I_0 .

3. PROOF OF THEOREM 1.4 AND COROLLARIES

In this section we prove Theorem 1.4, Corollary 1.5 and Corollary 1.6.

First we recall the following theorem proved by Birkar, which plays a crucial role in the proof of Theorem 1.4.

Theorem 3.1 (cf. [B1, Corollary 1.7]). *Fix a positive integer d , and assume Conjecture 1.2 _{$\leq d-1$} . Let (X, Δ) be a d -dimensional projective log canonical pair such that $K_X + \Delta \sim_{\mathbb{R}} D$ for an effective \mathbb{R} -divisor D . Then Conjecture 1.2 holds for (X, Δ) .*

The following lemma is known to the experts, but we write details of proof for reader's convenience.

Lemma 3.2. *Conjecture 1.3 _{n} implies Conjecture 1.3 _{$\leq n$} .*

Proof. Assume Conjecture 1.3 _{n} and pick any $d \leq n$. Let X be a smooth projective variety of dimension d such that K_X is pseudo-effective. Let W be the product of X and an $(n - d)$ -dimensional abelian variety, and let $f : W \rightarrow X$ be the projection. Then $K_W = f^*K_X$ and K_W is pseudo-effective. Since we assume Conjecture 1.3 _{n} , Conjecture 1.3 holds for W , and therefore Conjecture 1.3 holds for X . So we are done. \square

From now on we prove Theorem 1.4. We fix n in Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.2 we may assume Conjecture 1.3 _{$\leq n$} . Pick any positive integer $d \leq n$. We prove Conjecture 1.1 _{$\leq d$} and Conjecture 1.2 _{$\leq d$} at once by the induction on d . If we can prove this then

Theorem 1.4 immediately follows. From now on we assume Conjecture 1.1_{≤d-1} and Conjecture 1.2_{≤d-1}, and let (X, Δ) be a d -dimensional lc pair. By Theorem 3.1, we only have to prove that Conjecture 1.1 holds for (X, Δ) . By taking a dlt blow-up, we can assume that (X, Δ) is \mathbb{Q} -factorial dlt. We can write $\Delta = S + B$, where S is the reduced part of Δ and $B = \Delta - S$. Then we have following two cases.

Case 1. $S \neq 0$ and $\tau(X, B; S) = 1$, where $\tau(X, B; S)$ is the pseudo-effective threshold of S with respect to (X, B) .

Case 2. $S \neq 0$ and $\tau(X, B; S) < 1$, or $S = 0$.

Proof of Case 1. We prove it with several steps.

Step 1. From this step to Step 5, we prove that Conjecture 1.2 holds for (X, Δ) .

We run the $(K_X + \Delta)$ -MMP with scaling of an ample divisor H

$$(X, \Delta) \dashrightarrow \cdots \dashrightarrow (X^i, \Delta_{X^i}) \dashrightarrow \cdots$$

Then for any i , the birational map $X \dashrightarrow \cdots \dashrightarrow X^i$ is also a finitely many steps of the $(K_X + tS + B)$ -MMP for any $t < 1$ sufficiently close to one. Since $K_X + tS + B$ is not pseudo-effective by hypothesis, we see that $S_{X^i} \neq 0$ and $\tau(X^i, B_{X^i}; S_{X^i}) = 1$ for any i . Therefore we can replace (X, Δ) with (X^i, Δ_{X^i}) for some $i \gg 0$ and we may assume that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$.

Step 2. Fix $A \geq 0$ a general ample divisor such that $(X, \Delta + A)$ is lc, $(X, B + A)$ is klt and $(1/2)A + K_X + B$ and $(1/2)A + S$ are both nef. Then

$$K_X + tS + B + A = \left(\frac{1}{2}A + K_X + B\right) + t\left(\frac{1}{2}A + S\right) + \frac{1}{2}(1-t)A$$

is nef for any $0 \leq t \leq 1$. Let $\tau_t = \tau(X, tS + B; A)$ be the pseudo-effective threshold of A with respect to $(X, tS + B)$ for any $0 \leq t < 1$. By construction we have $0 < \tau_t \leq 1$ for any t . In this step we prove that there is $0 < \epsilon < 1$ such that the divisor

$$\begin{aligned} & K_X + (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A \\ &= (1 - t)(K_X + \Delta) + t(K_X + \epsilon S + B + \tau_\epsilon A) \end{aligned}$$

is not big for any $0 \leq t \leq 1$.

The idea is similar to [H, Step 2 and Step 3 in the proof of Proposition 5.3]. Let $\{t_k\}_{k \geq 1}$ be a strictly increasing infinite sequence of positive real numbers such that $t_k < 1$ for any k and $\lim_{k \rightarrow \infty} t_k = 1$. For each k , we run the $(K_X + t_k S + B)$ -MMP with scaling of A . Then we get a Mori fiber space $(X, t_k S + B) \dashrightarrow (X_k, t_k S_{X_k} + B_{X_k}) \rightarrow Z_k$. By the basic

property of the log MMP with scaling, $K_{X_k} + t_k S_{X_k} + B_{X_k} + \tau_{t_k} A_{X_k}$ is trivial over Z_k .

We consider the set

$$\{\mu \in \mathbb{R}_{\geq 0} \mid \mu = \text{lct}(X_k, B_{X_k}; S_{X_k}) \text{ for some } k \geq 1\}.$$

By construction we have $\text{lct}(X_k, B_{X_k}; S_{X_k}) \geq t_k$ for any k . Moreover the coefficients of B_{X_k} and S_{X_k} belong to a finite set which does not depend on k . Since $\lim_{k \rightarrow \infty} t_k = 1$, by the ACC for log canonical thresholds (cf. Theorem 2.7), there are infinitely many indices k such that $\text{lct}(X_k, B_{X_k}; S_{X_k}) = 1$. In this way, by replacing $\{t_k\}_{k \geq 1}$ with its subsequence, we may assume that (X_k, Δ_{X_k}) is lc for any k . Moreover we may assume that $\dim Z_k$ is constant for any k by replacing $\{t_k\}_{k \geq 1}$ with its subsequence again.

By construction we see that S_{X_k} is ample over Z_k . For any k , let ν_k be the real number such that $K_{X_k} + \nu_k S_{X_k} + B_{X_k} \equiv_{Z_k} 0$. Then we can check that $t_k < \nu_k \leq 1$. Let F_k be the general fiber of $X_k \rightarrow Z_k$. Then $\dim F_k$ is constant for any k and we have $K_{F_k} + \nu_k S_{F_k} + B_{F_k} \equiv 0$, where S_{F_k} (resp. B_{F_k}) is the restriction of S_{X_k} (resp. B_{X_k}) to F_k . Note that S_{F_k} is ample by construction. We consider the set

$$\mathcal{I} = \{\nu \in \mathbb{R}_{\geq 0} \mid K_{F_k} + \nu S_{F_k} + B_{F_k} \equiv 0 \text{ for some } k \geq 1\}.$$

It is clear that $\mathcal{I} \supset \{\nu_k\}_{k \geq 1}$. On the other hand we see that ν_k is the unique number satisfying $K_{F_k} + \nu_k S_{F_k} + B_{F_k} \equiv 0$ since S_{F_k} is ample. Thus we see that $\mathcal{I} = \{\nu_k\}_{k \geq 1}$. We can easily check that the coefficients of S_{F_k} and B_{F_k} belong to a finite set which does not depend on k . Since $t_k < \nu_k \leq 1$ and $\lim_{k \rightarrow \infty} t_k = 1$, by the ACC for numerically trivial pairs (cf. Theorem 2.8), \mathcal{I} must contain one. Then $\nu_k = 1$ for some k . In this way we can find an index k such that $K_{X_k} + \Delta_{X_k}$ is numerically trivial over Z_k and therefore $K_{X_k} + \Delta_{X_k}$ is trivial over Z_k .

Set $\epsilon = t_k$ for this k . Then

$$(1 - t)(K_{X_k} + \Delta_{X_k}) + t(K_{X_k} + \epsilon S_{X_k} + B_{X_k} + \tau_\epsilon A_{X_k})$$

is trivial over Z_k for any $0 \leq t \leq 1$. Since $\dim Z_k < \dim X_k$, we see that $K_X + (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A$ is not big. Note that $0 < \tau_\epsilon \leq 1$.

Step 3. Set $G = (1 - \epsilon)S - \tau_\epsilon A$. Then $(X, \Delta - tG)$ is klt and $\Delta - tG$ is big for any $0 < t \leq 1$ because $\Delta - tG = (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A$. In this step we show that there is an infinite sequence $\{a_k\}_{k \geq 1}$ of positive real numbers such that

- (i) $a_k < 1$ for any k and $\lim_{k \rightarrow \infty} a_k = 0$, and
- (ii) there is a finitely many steps of the $(K_X + \Delta - a_k G)$ -MMP to a good minimal model

$$(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$$

such that $(X'_k, \Delta_{X'_k})$ is lc and there is a contraction $X'_k \rightarrow Y_k$ to a normal projective variety Y_k such that

- (ii-a) $\dim Y_k < \dim X'_k$, and
- (ii-b) $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is \mathbb{R} -linearly equivalent to the pullback of an ample \mathbb{R} -divisor on Y_k and $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$.

The idea is similar to [H, Step 1 and Step 5 in the proof of Proposition 5.1]. Pick a strictly decreasing infinite sequence $\{t_k\}_{k \geq 1}$ of positive real numbers such that $t_k < 1$ for any k and $\lim_{k \rightarrow \infty} t_k = 0$. We run the $(K_X + \Delta - t_k G)$ -MMP with scaling of an ample divisor and we get a good minimal model $(X, \Delta - t_k G) \dashrightarrow (X_k, \Delta_{X_k} - t_k G_{X_k})$ by [BCHM, Corollary 1.4.2]. Next we pick positive real numbers $t'_k < t_k$ such that t'_k is sufficiently close to t_k for any k . Then we can assume that $X \dashrightarrow X_k$ is also a finitely many steps of the $(K_X + \Delta - uG)$ -MMP for any $t'_k \leq u \leq t_k$. We run the $(K_{X_k} + \Delta_{X_k} - t'_k G_{X_k})$ -MMP with scaling of an ample divisor and we get a good minimal model $(X_k, \Delta_{X_k} - t'_k G_{X_k}) \dashrightarrow (X'_k, \Delta_{X'_k} - t'_k G_{X'_k})$ by [BCHM, Corollary 1.4.2]. Since $K_{X_k} + \Delta_{X_k} - t_k G_{X_k}$ is semi-ample and by the standard arguments of the length of extremal rays (cf. [B1, Proposition 3.2]), we see that $K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}$ is semi-ample. By construction we also see that the composition of the birational maps $X \dashrightarrow X_k \dashrightarrow X'_k$ is a finitely many steps of the $(K_X + \Delta - uG)$ -MMP for any $t'_k \leq u < t_k$. Moreover, since

$$\text{lct}(X'_k, B_{X'_k}; S_{X'_k}) \geq 1 - t'_k > 1 - t_k$$

and $\lim_{k \rightarrow \infty} t_k = 0$, we can find infinitely many indices k such that $(X'_k, \Delta_{X'_k})$ is lc by Theorem 2.7 (see Step 2). By replacing $\{t_k\}_{k \geq 1}$ with its subsequence, we may assume that $(X'_k, \Delta_{X'_k})$ is lc for any k .

By construction $K_{X'_k} + \Delta_{X'_k} - t'_k G_{X'_k}$ and $K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}$ are semi-ample. Pick real numbers a_k and a'_k satisfying $t'_k < a'_k < a_k < t_k$. Then $X \dashrightarrow X'_k$ is a finitely many steps of the $(K_X + \Delta - a_k G)$ -MMP, and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ and $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}$ are semi-ample. Let $f_k : X'_k \rightarrow Y_k$ (resp. $f'_k : X'_k \rightarrow Y'_k$) be a contraction such that $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ (resp. $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}$) is \mathbb{R} -linearly equivalent to the pullback of an ample divisor on Y_k (resp. Y'_k). Then we see that $Y_k \simeq Y'_k$. Indeed, let C be any curve on X'_k . Then

$$\begin{aligned} & C \text{ is contracted by } f_k \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}) = C \cdot (K_{X'_k} + \Delta_{X'_k} - t'_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \text{ is contracted by } f'_k. \end{aligned}$$

Thus $Y_k \simeq Y'_k$. Then we have $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $a_k \neq a'_k$, we have $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. From Step 2, we also have that $K_X + \Delta - a_k G$ is not big for any k . Then $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is not big, and hence we see that $\dim Y_k < \dim X'_k$.

By the above arguments, the sequence $\{a_k\}_{k \geq 1}$ satisfies the condition (ii) stated at the start of this step. Since $a_k < t_k$ and $\lim_{k \rightarrow \infty} t_k = 0$, $\{a_k\}_{k \geq 1}$ satisfies the condition (i) stated at the start of this step. In this way we can find $\{a_k\}_{k \geq 1}$ satisfying the conditions (i) and (ii).

Step 4. In this step we prove that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k , where $(X'_k, \Delta_{X'_k})$ was constructed in Step 3. By Theorem 3.1, we may show that Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$ for any k . In this step we fix k .

Since $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$, we have $G_{X'_k} = (1 - \epsilon)S_{X'_k} - \tau_\epsilon A_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $A_{X'_k}$ is big, we see that $S_{X'_k}$ is big over Y_k . Therefore $S_{X'_k} \neq 0$ and some component of $S_{X'_k}$ dominates Y_k because $\dim Y_k < \dim X_k$ by the condition (ii-a) in Step 3. Let $f : (V, \Gamma) \rightarrow (X'_k, \Delta_{X'_k})$ be a dlt blow-up and let T be a component of $f_*^{-1}S_{X'_k}$ dominating Y_k . Then we have $K_V + \Gamma \sim_{\mathbb{R}, Y_k} 0$. Let M be an \mathbb{R} -divisor on Y_k such that $K_V + \Gamma$ is \mathbb{R} -linearly equivalent to the pullback of M . Then M is pseudo-effective. By the adjunction $(T, \text{Diff}(\Gamma - T))$ is dlt, and $K_T + \text{Diff}(\Gamma - T)$ is pseudo-effective. Since we assume Conjecture 1.1 $_{\leq d-1}$, Conjecture 1.1 holds for $(T, \text{Diff}(\Gamma - T))$. Then there is an effective \mathbb{R} -divisor E on Y_k such that $M \sim_{\mathbb{R}} E$. Therefore Conjecture 1.1 holds for (V, Γ) , and hence Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$.

In this way we see that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$.

Step 5. In this step we show that Conjecture 1.2 holds for (X, Δ) .

We keep the track of [H, Step 4 in the proof of Theorem 5.1]. By replacing $\{a_k\}_{k \geq 1}$ with its subsequence, we can assume that X'_i and X'_j are isomorphic in codimension one for any i and j . Indeed, any prime divisor P contracted by the birational map $X \dashrightarrow X'_k$ is a component of $N_\sigma(K_X + \Delta - a_k G)$. But since we have

$$N_\sigma(K_X + \Delta - a_k G) \leq (1 - a_k)N_\sigma(K_X + \Delta) + a_k N_\sigma(K_X + \Delta - G),$$

P is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - G)$, which does not depend on k . Thus we can assume that X'_i and X'_j are isomorphic in codimension one by replacing $\{a_k\}_{k \geq 1}$ with its subsequence.

By Step 4, $(X'_1, \Delta_{X'_1})$ has a log minimal model. Therefore, by [B3, Theorem 4.1 (iii)], we can run the $(K_{X'_1} + \Delta_{X'_1})$ -MMP with scaling of an ample divisor and get a log minimal model $(X'_1, \Delta_{X'_1}) \dashrightarrow (X'', \Delta_{X''})$.

Then we can check that $(X'', \Delta_{X''} - tG_{X''})$ is klt and $\Delta_{X''} - tG_{X''}$ is big for any sufficiently small $t > 0$. Fix a sufficiently small positive real number $t \ll a_1$. By [BCHM, Corollary 1.4.2] and running the $(K_{X''} + \Delta_{X''} - tG_{X''})$ -MMP with scaling of an ample divisor, we can get a log minimal model $(X'', \Delta_{X''} - tG_{X''}) \dashrightarrow (X''', \Delta_{X'''} - tG_{X'''})$. Since $t > 0$ is sufficiently small, by the standard arguments of the length of extremal rays (cf. [B1, Proposition 3.2]), $K_{X'''} + \Delta_{X'''}$ is nef. Now we get the following sequence of birational maps

$$X \dashrightarrow X'_1 \dashrightarrow X'' \dashrightarrow X'''$$

where $X \dashrightarrow X'_1$ (resp. $X'_1 \dashrightarrow X''$, $X'' \dashrightarrow X'''$) is a finitely many steps of the $(K_X + \Delta - a_1G)$ -MMP (resp. the $(K_{X'_1} + \Delta_{X'_1})$ -MMP, the $(K_{X''} + \Delta_{X''} - tG_{X''})$ -MMP) to a log minimal model.

We can show that X'_1 and X''' are isomorphic in codimension one. To see this, we may show that $X'_1 \dashrightarrow X''$ and $X'' \dashrightarrow X'''$ contain only flips. Recall that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$, which is stated in Step 1. Since $X \dashrightarrow X'_1$ is a birational contraction, $K_{X'_1} + \Delta_{X'_1} + \delta H_{X'_1}$ is movable for any sufficiently small $\delta > 0$. Then $N_\sigma(K_{X'_1} + \Delta_{X'_1}) = 0$ and thus $X'_1 \dashrightarrow X''$ contains only flips. Furthermore we see that $N_\sigma(K_{X''} + \Delta_{X''} - a_1G_{X''}) = 0$ since $K_{X'_1} + \Delta_{X'_1} - a_1G_{X'_1}$ is semi-ample, which is the condition (ii) in Step 3. Now we have $N_\sigma(K_{X''} + \Delta_{X''}) = 0$, which follows from that $K_{X''} + \Delta_{X''}$ is nef. From these facts we have

$$\begin{aligned} & N_\sigma(K_{X''} + \Delta_{X''} - tG_{X''}) \\ & \leq \left(1 - \frac{t}{a_1}\right) N_\sigma(K_{X''} + \Delta_{X''}) + \frac{t}{a_1} N_\sigma(K_{X''} + \Delta_{X''} - a_1G_{X''}) = 0 \end{aligned}$$

and hence $X'' \dashrightarrow X'''$ contains only flips. In this way we see that X'_1 and X''' are isomorphic in codimension one.

Since $\lim_{k \rightarrow \infty} a_k = 0$, we have $t \geq a_k$ for any $k \gg 0$. Then

$$K_{X'''} + \Delta_{X'''} - a_k G_{X'''}$$

is nef for any $k \gg 0$ because $K_{X'''} + \Delta_{X'''}$ and $K_{X'''} + \Delta_{X'''} - tG_{X'''}$ are nef. Moreover X''' and X'_k are isomorphic in codimension one since X'_k and X'_1 are isomorphic in codimension one and X'_1 and X''' are isomorphic in codimension one. We recall that $(X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ is a log minimal model of $(X, \Delta - a_k G)$, which is the condition (ii) in Step 3. From these facts, we see that $(X''', \Delta_{X'''} - a_k G_{X'''})$ is a log minimal model of $(X, \Delta - a_k G)$ for any $k \gg 0$. Let $p : W \rightarrow X$ and $q : W \rightarrow X'''$ be a common resolution of $X \dashrightarrow X'''$. Then for any $k \gg 0$, we have

$$p^*(K_X + \Delta - a_k G) - q^*(K_{X'''} + \Delta_{X'''} - a_k G_{X'''}) \geq 0.$$

By considering the limit $k \rightarrow \infty$, we have

$$p^*(K_X + \Delta) - q^*(K_{X'''} + \Delta_{X'''}) \geq 0.$$

Since $K_{X'''} + \Delta_{X'''}$ is nef, we see that $(X''', \Delta_{X'''})$ is a weak lc model of (X, Δ) . Therefore, by [B3, corollary 3.7], (X, Δ) has a log minimal model.

Step 6. Finally we prove that Conjecture 1.1 holds for (X, Δ) . By running the $(K_X + \Delta)$ -MMP with scaling of an ample divisor and replacing (X, Δ) with the resulting log minimal model, we can assume that $K_X + \Delta$ is nef. Note that after this process $S \neq 0$ and the equation $\tau(X, B; S) = 1$ still holds. Therefore $K_X + \Delta - tS$ is not pseudo-effective for any $t > 0$. Pick a sufficiently small positive real number t and run the $(K_X + \Delta - tS)$ -MMP with scaling of an ample divisor. Then we get a Mori fiber space

$$(X, \Delta - tS) \dashrightarrow (X', \Delta_{X'} - tS_{X'}) \rightarrow Z.$$

Moreover, since t is sufficiently small, $K_{X'} + \Delta_{X'}$ is trivial over Z and Conjecture 1.1 for (X, Δ) is equivalent to Conjecture 1.1 for $(X', \Delta_{X'})$ (see [B1, Proposition 3.2]). We also see that there is a component of $S_{X'}$ dominating Z because $S_{X'}$ is ample over Z . By the same arguments as in Step 4 we can prove that Conjecture 1.1 holds for $(X', \Delta_{X'})$ with the adjunction and Conjecture 1.1 _{$\leq d-1$} . Thus Conjecture 1.1 holds for (X, Δ_X) and so we are done. \square

Proof of Case 2. In this case we can assume that (X, Δ) is klt since we only have to prove that Conjecture 1.1 holds for (X, Δ) . Taking a log resolution of (X, Δ) , we can assume that X is smooth. We put $\tau = \tau(X, 0; \Delta)$. Then we may assume that $\Delta \neq 0$ and $\tau > 0$ because otherwise Conjecture 1.1 for (X, Δ) is obvious from Conjecture 1.3 _{$\leq n$} . Moreover we may assume that $\tau = 1$ by replacing (X, Δ) with $(X, \tau\Delta)$.

We prove Case 2 with several steps. The proof is very similar to the proof of Case 1 except Step 4. In the rest of the proof we will use the fact that (X, Δ) is \mathbb{Q} -factorial klt but we will not use the assumption that X is smooth.

Step 1. From this step to Step 5, we prove that Conjecture 1.2 holds for (X, Δ) .

We run the $(K_X + \Delta)$ -MMP with scaling of an ample divisor H

$$(X, \Delta) \dashrightarrow \cdots \dashrightarrow (X^i, \Delta_{X^i}) \dashrightarrow \cdots.$$

Then for any i , the birational map $X \dashrightarrow \cdots \dashrightarrow X^i$ is also a finitely many steps of the $(K_X + t\Delta)$ -MMP for any $t < 1$ sufficiently close to

one. Since $K_X + t\Delta$ is not pseudo-effective by hypothesis, we see that $\Delta_{X^i} \neq 0$ and $\tau(X^i, 0; \Delta_{X^i}) = 1$ for any i . Therefore we can replace (X, Δ) with (X^i, Δ_{X^i}) for some $i \gg 0$ and we may assume that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$.

Step 2. Fix $A \geq 0$ a general ample \mathbb{R} -divisor such that $(X, \Delta + A)$ is klt and $(1/2)A + K_X$ and $(1/2)A + \Delta$ are both nef. Then

$$K_X + t\Delta + A = \left(\frac{1}{2}A + K_X\right) + t\left(\frac{1}{2}A + \Delta\right) + \frac{1}{2}(1-t)A$$

is nef for any $0 \leq t \leq 1$. We put $\tau_t = \tau(X, t\Delta; A)$ for any $0 \leq t < 1$. By construction we have $0 < \tau_t \leq 1$ for any t . In this step we prove that there is $0 < \epsilon < 1$ such that the divisor

$$K_X + (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A = (1 - t)(K_X + \Delta) + t(K_X + \epsilon\Delta + \tau_\epsilon A)$$

is not big for any $0 \leq t \leq 1$.

Let $\{t_k\}_{k \geq 1}$ be a strictly increasing infinite sequence of positive real numbers such that $t_k < 1$ for any k and $\lim_{k \rightarrow \infty} t_k = 1$. For each k , run the $(K_X + t_k\Delta)$ -MMP with scaling of A . Then we get a Mori fiber space $(X, t_k\Delta) \dashrightarrow (X_k, t_k\Delta_{X_k}) \rightarrow Z_k$. By the basic property of the log MMP with scaling, $K_{X_k} + t_k\Delta_{X_k} + \tau_{t_k}A_{X_k}$ is trivial over Z_k . Now we carry out the same arguments as in Step 2 in the proof of Case 1, and we can find an index k such that (X_k, Δ_{X_k}) is lc and $K_{X_k} + \Delta_{X_k}$ is numerically trivial over Z_k by the ACC for log canonical thresholds (cf. Theorem 2.7) and the ACC for numerically trivial pairs (cf. Theorem 2.8). Set $\epsilon = t_k$ for this k . Then

$$(1 - t)(K_{X_k} + \Delta_{X_k}) + t(K_{X_k} + \epsilon\Delta_{X_k} + \tau_\epsilon A_{X_k})$$

is trivial over Z_k for any $0 \leq t \leq 1$. Since $\dim Z_k < \dim X_k$, we see that $K_X + (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A$ is not big. Note that $0 < \tau_\epsilon \leq 1$.

Step 3. Set $G = (1 - \epsilon)\Delta - \tau_\epsilon A$. Then $(X, \Delta - tG)$ is klt and $\Delta - tG$ is big for any $0 < t \leq 1$ because $\Delta - tG = (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A$. In this step we show that there is an infinite sequence $\{a_k\}_{k \geq 1}$ of positive real numbers such that

- (i) $a_k < 1$ for any k and $\lim_{k \rightarrow \infty} a_k = 0$, and
- (ii) there is a finitely many steps of the $(K_X + \Delta - a_k G)$ -MMP to a good minimal model

$$(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$$

such that $(X'_k, \Delta_{X'_k})$ is lc and there is a contraction $X'_k \rightarrow Y_k$ to a normal projective variety Y_k such that

- (ii-a) $\dim Y_k < \dim X'_k$, and

(ii-b) $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is \mathbb{R} -linearly equivalent to the pullback of an ample \mathbb{R} -divisor on Y_k and $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$.

Pick a strictly decreasing infinite sequence $\{t_k\}_{k \geq 1}$ of positive real numbers such that $t_k < 1$ for any k and $\lim_{k \rightarrow \infty} t_k = 0$. Then we can run the $(K_X + \Delta - t_k G)$ -MMP with scaling of an ample divisor and get a good minimal model $(X, \Delta - t_k G) \dashrightarrow (X_k, \Delta_{X_k} - t_k G_{X_k})$ by [BCHM, Corollary 1.4.2]. Next we pick positive real numbers $t'_k < t_k$ such that t'_k is sufficiently close to t_k for any k . Then we can assume that $X \dashrightarrow X_k$ is also a finitely many steps of the $(K_X + \Delta - uG)$ -MMP for any $t'_k \leq u \leq t_k$. We run the $(K_{X_k} + \Delta_{X_k} - t'_k G_{X_k})$ -MMP with scaling of an ample divisor and we get a good minimal model $(X_k, \Delta_{X_k} - t'_k G_{X_k}) \dashrightarrow (X'_k, \Delta_{X'_k} - t'_k G_{X'_k})$ by [BCHM, Corollary 1.4.2]. Since $K_{X_k} + \Delta_{X_k} - t_k G_{X_k}$ is semi-ample and by the standard arguments of the length of extremal rays (cf. [B1, Proposition 3.2]), we see that $K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}$ is semi-ample. By construction we also see that the composition of the birational maps $X \dashrightarrow X_k \dashrightarrow X'_k$ is a finitely many steps of the $(K_X + \Delta - uG)$ -MMP for any $t'_k \leq u < t_k$. Moreover, since

$$\text{lct}(X'_k, 0; \Delta_{X'_k}) \geq 1 - t'_k > 1 - t_k$$

and $\lim_{k \rightarrow \infty} t_k = 0$, we can find infinitely many indices k such that $(X'_k, \Delta_{X'_k})$ is lc by Theorem 2.7 (see Step 2 in the proof of Case 1). By replacing $\{t_k\}_{k \geq 1}$ with its subsequence, we may assume that $(X'_k, \Delta_{X'_k})$ is lc for any k .

By construction $K_{X'_k} + \Delta_{X'_k} - t'_k G_{X'_k}$ and $K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}$ are semi-ample. Pick real numbers a_k and a'_k satisfying $t'_k < a'_k < a_k < t_k$. Then $X \dashrightarrow X'_k$ is a finitely many steps of the $(K_X + \Delta - a_k G)$ -MMP, and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ and $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}$ are semi-ample. Let $f_k : X'_k \rightarrow Y_k$ (resp. $f'_k : X'_k \rightarrow Y'_k$) be a contraction such that $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ (resp. $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}$) is \mathbb{R} -linearly equivalent to the pullback of an ample divisor on Y_k (resp. Y'_k). Then we see that $Y_k \simeq Y'_k$. Indeed, let C be any curve on X'_k . Then

$$\begin{aligned} & C \text{ is contracted by } f_k \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - t_k G_{X'_k}) = C \cdot (K_{X'_k} + \Delta_{X'_k} - t'_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \cdot (K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k}) = 0 \\ \Leftrightarrow & C \text{ is contracted by } f'_k. \end{aligned}$$

Thus $Y_k \simeq Y'_k$. Then we have $K_{X'_k} + \Delta_{X'_k} - a'_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $a_k \neq a'_k$, we have $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$.

From Step 2, we also have that $K_X + \Delta - a_k G$ is not big for any k . Then $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is not big, and hence we see that $\dim Y_k < \dim X'_k$.

By the above arguments, the sequence $\{a_k\}_{k \geq 1}$ satisfies the condition (ii) stated at the start of this step. Since $a_k < t_k$ and $\lim_{k \rightarrow \infty} t_k = 0$, $\{a_k\}_{k \geq 1}$ satisfies the condition (i) stated at the start of this step. In this way we can find $\{a_k\}_{k \geq 1}$ satisfying the conditions (i) and (ii).

Step 4. In this step we prove that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k , where $(X'_k, \Delta_{X'_k})$ was constructed in Step 3. We note that (X, Δ) is klt but $(X'_k, \Delta_{X'_k})$ may not be klt. By Theorem 3.1, we only have to show that Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$ for any k . In this step we fix k .

If $(X'_k, \Delta_{X'_k})$ is klt, by applying Ambro's canonical bundle formula (cf. [FG, Corollary 3.2]) to $X'_k \rightarrow Y_k$ and since we assume Conjecture 1.1 $_{\leq d-1}$, Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$. Therefore we may assume that $(X'_k, \Delta_{X'_k})$ is not klt.

Let $f : (V, \Gamma) \rightarrow (X'_k, \Delta_{X'_k})$ be a dlt blow-up of $(X'_k, \Delta_{X'_k})$ and we write $\Gamma = S_V + B_V$, where S_V is the reduced part of Γ and $B_V = \Gamma - S_V$. Then $S_V \neq 0$ by our assumption. We may prove that Conjecture 1.1 holds for (V, Γ) . If $\tau(V, B_V; S_V) = 1$, then Conjecture 1.1 holds for (V, Γ) by Case 1. Therefore we may assume that $\tau(V, B_V; S_V) < 1$. Note that $K_V + \Gamma \sim_{\mathbb{R}, Y_k} 0$ by construction.

Since $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$, we have $G_{X'_k} = (1 - \epsilon)\Delta_{X'_k} - \tau_\epsilon A_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $A_{X'_k}$ is big, we see that $\Delta_{X'_k}$ is big over Y_k . Then Γ is also big over Y_k because Γ contains $f_*^{-1}\Delta_{X'_k}$ and all f -exceptional prime divisors. We pick sufficiently small positive real number $t < 1$ so that $\tau(V, B_V; S_V) \leq 1 - t$ and $\Gamma - tS_V$ is big over Y_k . Then $(V, \Gamma - tS_V)$ is klt and $K_V + \Gamma - tS_V$ is pseudo-effective. Moreover we see that $K_V + \Gamma - tS_V$ is not big over Y_k because $K_V + \Gamma - tS_V \sim_{\mathbb{R}, Y_k} -tS_V$ and $\dim Y_k < \dim V$. By construction we only have to prove that Conjecture 1.1 holds for $(V, \Gamma - tS_V)$.

We run the $(K_V + \Gamma - tS_V)$ -MMP over Y_k with scaling of an ample divisor. By [BCHM, Corollary 1.4.2], we get a good minimal model $(V, \Gamma - tS_V) \dashrightarrow (V', \Gamma_{V'} - tS_{V'})$ over Y_k , where $S_{V'}$ is the birational transform of S_V on V' . Then there is a contraction $V' \rightarrow \tilde{Y}$ to a normal projective variety \tilde{Y} over Y_k such that $K_{V'} + \Gamma_{V'} - tS_{V'} \sim_{\mathbb{R}, \tilde{Y}} 0$. We can check that $(V', \Gamma_{V'} - tS_{V'})$ is klt, and furthermore $\dim \tilde{Y} < \dim V'$ since $K_{V'} + \Gamma_{V'} - tS_{V'}$ is not big over Y_k . Therefore, applying Ambro's canonical bundle formula (cf. [FG, Corollary 3.2]) to $V' \rightarrow \tilde{Y}$ and since we assume Conjecture 1.1 $_{\leq d-1}$, Conjecture 1.1 holds for $(V', \Gamma_{V'} - tS_{V'})$.

Then Conjecture 1.1 holds for $(V, \Gamma - tS_V)$, and thus Conjecture 1.1 holds for (V, Γ_V) .

In this way we see that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k , and we complete this step.

Step 5. In this step we show that Conjecture 1.2 holds for (X, Δ) .

By replacing $\{a_k\}_{k \geq 1}$ with its subsequence, we can assume that X'_i and X'_j are isomorphic in codimension one for any i and j . Indeed, any prime divisor P contracted by the birational map $X \dashrightarrow X'_k$ is a component of $N_\sigma(K_X + \Delta - a_k G)$. But since we have

$$N_\sigma(K_X + \Delta - a_k G) \leq (1 - a_k)N_\sigma(K_X + \Delta) + a_k N_\sigma(K_X + \Delta - G),$$

P is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - G)$, which does not depend on k . Thus we can assume that X'_i and X'_j are isomorphic in codimension one by replacing $\{a_k\}_{k \geq 1}$ with its subsequence.

Since $(X'_1, \Delta_{X'_1})$ has a log minimal model, by [B3, Theorem 4.1 (iii)], we can run the $(K_{X'_1} + \Delta_{X'_1})$ -MMP with scaling of an ample divisor and get a log minimal model $(X'_1, \Delta_{X'_1}) \dashrightarrow (X'', \Delta_{X''})$. Then we can check that $(X'', \Delta_{X''} - tG_{X''})$ is klt and $\Delta_{X''} - tG_{X''}$ is big for any sufficiently small $t > 0$. Fix a sufficiently small positive real number $t \ll a_1$. By [BCHM, Corollary 1.4.2] and running the $(K_{X''} + \Delta_{X''} - tG_{X''})$ -MMP with scaling of an ample divisor, we can get a log minimal model $(X'', \Delta_{X''} - tG_{X''}) \dashrightarrow (X''', \Delta_{X'''} - tG_{X'''})$. Since $t > 0$ is sufficiently small, by the standard arguments of the length of extremal rays (cf. [B1, Proposition 3.2]), we see that $K_{X'''} + \Delta_{X'''}$ is nef. Now we get the following sequence of birational maps

$$X \dashrightarrow X'_1 \dashrightarrow X'' \dashrightarrow X'''$$

where $X \dashrightarrow X'_1$ (resp. $X'_1 \dashrightarrow X''$, $X'' \dashrightarrow X'''$) is a finitely many steps of the $(K_X + \Delta - a_1 G)$ -MMP (resp. the $(K_{X'_1} + \Delta_{X'_1})$ -MMP, the $(K_{X''} + \Delta_{X''} - tG_{X''})$ -MMP) to a log minimal model.

We can show that X'_1 and X''' are isomorphic in codimension one. To see this, we may show that $X'_1 \dashrightarrow X''$ and $X'' \dashrightarrow X'''$ contain only flips. Recall that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$, which is stated in Step 1 in this proof. Since $X \dashrightarrow X'_1$ is a birational contraction, $K_{X'_1} + \Delta_{X'_1} + \delta H_{X'_1}$ is movable for any sufficiently small $\delta > 0$. Then $N_\sigma(K_{X'_1} + \Delta_{X'_1}) = 0$ and thus $X'_1 \dashrightarrow X''$ contains only flips. Furthermore we see that $N_\sigma(K_{X''} + \Delta_{X''} - a_1 G_{X''}) = 0$ since $K_{X'_1} + \Delta_{X'_1} - a_1 G_{X'_1}$ is semi-ample, which is the condition (ii) in Step 3. Now we have $N_\sigma(K_{X''} + \Delta_{X''}) = 0$,

which follows from that $K_{X''} + \Delta_{X''}$ is nef. From these facts, we have

$$\begin{aligned} & N_\sigma(K_{X''} + \Delta_{X''} - tG_{X''}) \\ & \leq \left(1 - \frac{t}{a_1}\right) N_\sigma(K_{X''} + \Delta_{X''}) + \frac{t}{a_1} N_\sigma(K_{X''} + \Delta_{X''} - a_1 G_{X''}) = 0 \end{aligned}$$

and hence $X'' \dashrightarrow X'''$ contains only flips. In this way we see that X'_1 and X''' are isomorphic in codimension one.

Since $\lim_{k \rightarrow \infty} a_k = 0$, we have $t \geq a_k$ for any $k \gg 0$. Then

$$K_{X'''} + \Delta_{X'''} - a_k G_{X'''}$$

is nef for any $k \gg 0$ because $K_{X'''} + \Delta_{X'''}$ and $K_{X'''} + \Delta_{X'''} - tG_{X'''}$ are nef. Moreover X''' and X'_k are isomorphic in codimension one since X'_k and X'_1 are isomorphic in codimension one and X'_1 and X''' are isomorphic in codimension one. We recall that $(X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ is in particular a log minimal model of $(X, \Delta - a_k G)$, which is the condition (ii) in Step 3. From these facts, we see that $(X''', \Delta_{X'''} - a_k G_{X'''})$ is a log minimal model of $(X, \Delta - a_k G)$ for any $k \gg 0$. Let $p : W \rightarrow X$ and $q : W \rightarrow X'''$ be a common resolution of $X \dashrightarrow X'''$. Then for any $k \gg 0$ we have

$$p^*(K_X + \Delta - a_k G) - q^*(K_{X'''} + \Delta_{X'''} - a_k G_{X'''}) \geq 0.$$

By considering the limit $k \rightarrow \infty$, we have

$$p^*(K_X + \Delta) - q^*(K_{X'''} + \Delta_{X'''}) \geq 0.$$

Since $K_{X'''} + \Delta_{X'''}$ is nef, we see that $(X''', \Delta_{X'''})$ is a weak lc model of (X, Δ) . Therefore, by [B3, corollary 3.7], (X, Δ) has a log minimal model.

Step 6. Finally we prove that Conjecture 1.1 holds for (X, Δ) . By [B3, Theorem 4.1 (iii)] and running the $(K_X + \Delta)$ -MMP with scaling of an ample divisor, we get a log minimal model $(X, \Delta) \dashrightarrow (X', \Delta_{X'})$. Then we can check that $\Delta_{X'} \neq 0$ and $\tau(X', 0; \Delta_{X'}) = 1$. Therefore, by replacing (X, Δ) with $(X', \Delta_{X'})$, we can assume that $K_X + \Delta_X$ is nef. Then $K_X + (1 - t)\Delta$ is not pseudo-effective for any $t > 0$. Pick a sufficiently small positive real number t and run the $(K_X + (1 - t)\Delta)$ -MMP with scaling of an ample divisor. Then we get a Mori fiber space

$$(X, (1 - t)\Delta) \dashrightarrow (X', (1 - t)\Delta_{X'}) \rightarrow Z.$$

Moreover, since t is sufficiently small, $K_{X'} + \Delta_{X'}$ is trivial over Z and Conjecture 1.1 for (X, Δ) is equivalent to Conjecture 1.1 for $(X', \Delta_{X'})$ (see [B1, Proposition 3.2]). Now we can easily check that Conjecture 1.1 for $(X', \Delta_{X'})$ holds by Ambro's canonical bundle formula (cf. [FG, Corollary 3.2]) and Conjecture 1.1 _{$\leq d-1$} . So we are done.

□

Therefore we complete the proof of Theorem 1.4. □

Proof of Corollary 1.5. It immediately follows from Theorem 1.4. □

Proof of Corollary 1.6. It immediately follows from Theorem 1.4. □

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